

THE PADÉ INTERPOLATION METHOD APPLIED TO q -PAINLEVÉ EQUATIONS II (DIFFERENTIAL GRID VERSION)

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ABSTRACT. Recently we studied Padé interpolation problems of q -grid, related to q -Painlevé equations of type $E_7^{(1)}$, $E_6^{(1)}$, $D_5^{(1)}$, $A_4^{(1)}$ and $(A_2 + A_1)^{(1)}$. By solving those problems, we could derive evolution equations, scalar Lax pairs and determinant formulae of special solutions for the corresponding q -Painlevé equations. It is natural that the q -Painlevé equations were derived by the interpolation method of q -grid, but it may be interesting in terms of differential grid that the Padé interpolation method of differential grid (i.e. Padé approximation method) has been applied to the q -Painlevé equation of type $D_5^{(1)}$ by Y. Ikawa. In this paper we continue the above study and apply the Padé approximation method to the q -Painlevé equations of type $E_6^{(1)}$, $D_5^{(1)}$, $A_4^{(1)}$ and $(A_2 + A_1)^{(1)}$. Moreover determinant formulae of the special solutions for q -Painlevé equation of type $E_6^{(1)}$ are given in terms of the terminating q -Appell Lauricella function.

1. INTRODUCTION

In this paper we continue [13] and apply the Padé approximation method to the q -Painlevé equations of type $E_6^{(1)}$, $D_5^{(1)}$, $A_4^{(1)}$ and $(A_2 + A_1)^{(1)}$.

1.1. The background of discrete Painlevé equations.

In Sakai's theory [21] the discrete Painlevé equations were classified on the basis of rational surfaces connected to extended affine Weyl groups. There exist three types of discrete Painlevé equations in the classification: elliptic difference (e -), multiplicative difference (q -) and additive difference (d -). The discrete Painlevé equations of q -difference type are classified as follows:

$$E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow \underset{(q-P_{VI})}{D_5^{(1)}} \rightarrow \underset{(q-P_V)}{A_4^{(1)}} \rightarrow \underset{(q-P_{IV}, q-P_{III})}{(A_2 + A_1)^{(1)}} \rightarrow \underset{(q-P_{II})}{(A_1 + A_1')^{(1)}} \rightarrow \underset{(q-P_I)}{A_1^{(1)}} \xrightarrow{\mathbb{Z}} \mathcal{D}_6$$

Here $A \rightarrow B$ means that B is obtained from A by degeneration.

1.2. The background of the Padé method.

Padé approximation/interpolation are closely related to Painlevé/Garnier equations. The Padé method is a method for giving Painlevé equations, scalar Lax pairs and determinant formulae of special solutions simultaneously, by starting from suitable problems of Padé approximation (of differential grid)/interpolation (of difference grid). In [30] Y. Yamada has applied the Padé

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method to continuous Painlevé equations of type P_{VI} , P_V , P_{IV} and Garnier system by using differential grid (i.e. Padé approximation).

The Padé method for discrete Painlevé equations has been applied to the following types:

	$e-E_8^{(1)}$	$q-E_8^{(1)}$	$q-E_7^{(1)}$	$q-E_6^{(1)}$	$q-D_5^{(1)}$	$q-A_4^{(1)}$	$q-(A_2 + A_1)^{(1)}$
(1.1)	[17]	[33]	[13]	[4, 13]	[4]	[13]	[13]
	grid	elliptic	q -quadric	q	q	differential	q

It is natural that the continuous/discrete Painlevé equations were derived by the interpolation of differential/difference grid respectively. Here it may be interesting to note that the Padé approximation method of differential grid (i.e. Padé approximation) has been applied to the type $q-D_5^{(1)}$ in [4]. In this paper differential grid is applied to type $q-E_6^{(1)}$, $q-D_5^{(1)}$, $q-A_4^{(1)}$ and $q-(A_2 + A_1)^{(1)}$.

Remark 1. On the key points of the Padé method

There are two key points to apply the Padé approximation/interpolation method [4, 13, 17, 33]. The first key point is the appropriate choice of approximated/interpolated functions (see Table (2.2) and Remark 4). The second key point is to consider two linear q -difference three term relations (2.6) satisfied by the error terms of the Padé approximation/interpolation problems. Then the error terms can be expressed in terms of special solutions of q -Painlevé equations. Therefore the q -difference relations are the main subject in our study, and they naturally give the evolution equations, the Lax pairs and the special solutions for the corresponding q -Painlevé equations. \square

Remark 2. On a connection between the Padé method and the theory of semiclassical orthogonal polynomials

The connection between semiclassical orthogonal polynomials (classical orthogonal polynomials related to a suitable weight function) and Painlevé/Garnier systems has been demonstrated in [11]. It has been shown that coefficients of three term recurrence relations, satisfied by several semiclassical orthogonal polynomials, can be expressed in terms of solutions of Painlevé/Garnier systems (see [1, 15, 18, 26, 27, 28, 29] for example). Thus there exists a close connection between the Padé method and the theory of semiclassical orthogonal polynomials. Namely, using both approaches, we can obtain the evolution equations, the Lax pairs and the special solutions for the corresponding Painlevé/Garnier systems. (The theory of semiclassical orthogonal polynomials is more general and the Padé method is simpler. For example their relation was briefly proved in [30].) \square

1.3. The purpose and the organization of this paper.

The purpose of this paper is to apply the Padé approximation method to type $q-E_6^{(1)}$, $q-D_5^{(1)}$, $q-A_4^{(1)}$ and $q-(A_2 + A_1)^{(1)}$. As the main results given in Section 3, the following items are presented for each type.

- (a) Setting of the Padé approximation problem,
- (b) Contiguity relations,

- (c) The Painlevé equation,
- (d) The Lax pair,
- (e) Special solutions.

This paper is organized as follows: In Section 2 we explain the Padé approximation method applied to the q -Painlevé equations, namely the methods for the items (a)–(e) above. In Section 3 we present these main results for type q - $E_6^{(1)}$, q - $D_5^{(1)}$, q - $A_4^{(1)}$ and q -($A_2 + A_1$) $^{(1)}$. In Section 4 we give a summary and discuss some future problems.

2. PADÉ APPROXIMATION METHOD OF DIFFERENTIAL GRID

In this section we explain the methods for deriving the items (a)–(e) in the main results given in Section 3. These contents of the items (b)–(d) below (i.e. Subsection 2.2, 2.3 and 2.4) are almost the same as the items (b)–(d) in Section 2 of [13].

2.1. (a) Setting of the Padé approximation problem.

Let us consider the following approximation problem (of differential grid):

For a given function $Y(x)$, we look for functions $P_m(x)$ and $Q_n(x)$ which are polynomials of degree m and $n \in \mathbb{Z}_{\geq 0}$, satisfying the approximation condition

$$(2.1) \quad Y(x) \equiv \frac{P_m(x)}{Q_n(x)} \pmod{x^{m+n+1}}.$$

We call this problem the "Padé approximation problem (of differential grid)". Then the function $Y(x)$ is called the "generating function" (because $Y(x)$ generates the coefficients p_k in power series (2.13) in the item (e) below, i.e. Subsection 2.5), and the polynomials $P_m(x)$ and $Q_n(x)$ are called "approximating polynomials" respectively. The explicit expressions of the polynomials $P_m(x)$ and $Q_n(x)$, which are used in the computations for the item (e) above, are given in the formulae (2.14) and (2.20) (see the item (e) below).

Remark 3. On the common normalization factor of the polynomials $P_m(x)$ and $Q_n(x)$

The common normalization factor of the approximating polynomials $P_m(x)$ and $Q_n(x)$ is not determined by the condition (2.1). However this normalization factor is not essential for our arguments, i.e. the main results in Section 3 (see Remark 5). \square

Fix a complex parameter q ($0 < |q| < 1$). Let $a_i, b_i \in \mathbb{C}^\times$ be complex parameters. We establish the approximation problems (2.1) by specifying the generating functions $Y(x)$ as follows:

$$(2.2) \quad \begin{array}{c|c|c|c|c} & q-E_6^{(1)} & q-D_5^{(1)} & q-A_4^{(1)} & q-(A_2 + A_1)^{(1)} \\ \hline Y(x) & \prod_{i=1}^3 \frac{(a_i x; q)_\infty}{(b_i x; q)_\infty} \\ & \frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1 & \prod_{i=1}^2 \frac{(a_i x; q)_\infty}{(b_i x; q)_\infty} & \frac{(a_1 x, a_2 x; q)_\infty}{(b_1 x; q)_\infty} & (a_1 x, a_2 x; q)_\infty \\ \hline q\text{-HGF} & {}_3\varphi_2 & {}_2\varphi_1 & {}_2\varphi_1 & {}_1\varphi_1 \end{array}$$

Here $\frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1$ is a constraint for the parameters in the case $q-E_6^{(1)}$, and the q -shifted factorials are defined by

$$(2.3) \quad (a_1, a_2, \dots, a_i; q)_j := \prod_{k=0}^{j-1} (1 - a_1 q^k)(1 - a_2 q^k) \cdots (1 - a_i q^k)$$

and the q -HGFs (i.e. the q -hypergeometric functions [2]) defined by

$$(2.4) \quad {}_k\varphi_l \left(\begin{matrix} a_1, & \dots, & a_k \\ b_1, & \dots, & b_l \end{matrix}; q, x \right) := \sum_{s=0}^{\infty} \frac{(a_1, \dots, a_k; q)_s}{(b_1, \dots, b_l; q)_s} [(-1)^s q^{\binom{s}{2}}]^{1+l-k} x^s$$

with $\binom{s}{2} = s(s-1)/2$.

The hypergeometric solutions to the q -Painlevé equations of type $E_6^{(1)}$, $D_5^{(1)}$, $A_4^{(1)}$ and $(A_2 + A_1)^{(1)}$ were given in terms of the q -hypergeometric functions ${}_3\varphi_2$, ${}_2\varphi_1$, ${}_2\varphi_1$, and ${}_1\varphi_1$ respectively in [6]. The functions $Y(x)$ in Table (2.2) generate the coefficients p_k in power series (2.13) in terms of the terminating q -hypergeometric functions ${}_3\varphi_2$ (3.13) (see Remark 7), ${}_2\varphi_1$ (3.25), ${}_2\varphi_1$ (3.36), and ${}_1\varphi_1$ (3.47).

Remark 4. On the choice of the generating functions $Y(x)$

One may wonder how the generating functions $Y(x)$ are appropriately chosen. However there is no theoretical choice of the functions $Y(x)$ in the Padé approximation method as far as we know. We chose the functions $Y(x)$ to type $E_6^{(1)}$, $A_4^{(1)}$ and $(A_2 + A_1)^{(1)}$ by a extension and reductions from the generating function to type $D_5^{(1)}$ given in [4]. \square

Let us consider yet another Padé problem where some parameters a_i, b_i, m and n in the generating functions $Y(x)$ are shifted. The parameter shift operators T are given as follows:

	parameter
$q-E_6^{(1)}$	$(a_1, a_2, a_3, b_1, b_2, b_3, m, n) \mapsto (qa_1, a_2, a_3, qb_1, b_2, b_3, m, n)$
$q-D_5^{(1)}$	$(a_1, a_2, b_1, b_2, m, n) \mapsto (qa_1, a_2, qb_1, b_2, m, n)$
$q-A_4^{(1)}$	$(a_1, a_2, b_1, m, n) \mapsto (qa_1, a_2, qb_1, m, n)$
$q-(A_2 + A_1)^{(1)}$	$(a_1, a_2, m, n) \mapsto (qa_1, a_2, m, n)$

Here the operators T are called the "time evolutions", because they specify the directions of the time evolutions for q -Painlevé equations.

2.2. (b) Contiguity relations.

Let us consider two linear three term relations: $L_2(x) = 0$ between $y(x), y(qx), \bar{y}(x)$ and $L_3(x) = 0$ between $y(x), \bar{y}(x), \bar{y}(x/q)$ satisfied by fundamental solutions $y(x) = P_m(x), Y(x)Q_n(x)$, where L_2 and L_3 are given as expressions

$$(2.6) \quad \begin{aligned} L_2(x) &\propto \begin{vmatrix} y(x) & y(qx) & \bar{y}(x) \\ P_m(x) & P_m(qx) & \bar{P}_m(x) \\ Y(x)Q_n(x) & Y(qx)Q_n(qx) & \bar{Y}(x)\bar{Q}_n(x) \end{vmatrix}, \\ L_3(x) &\propto \begin{vmatrix} y(x) & \bar{y}(x) & \bar{y}(x/q) \\ P_m(x) & \bar{P}_m(x) & \bar{P}_m(x/q) \\ Y(x)Q_n(x) & \bar{Y}(x)\bar{Q}_n(x) & \bar{Y}(x/q)\bar{Q}_n(x/q) \end{vmatrix}. \end{aligned}$$

Then the linear relations $L_2 = 0$ and $L_3 = 0$ are called the "contiguity relations", and the contiguity relations are the main subject in our study. Here for any object F the corresponding shifts are denoted as $\overline{F} := T(F)$ and $\underline{F} := T^{-1}(F)$, and the shift operator T acts on parameters given in Table (2.5).

We show the method of computation of the contiguity relations $L_2 = 0$ and $L_3 = 0$.

Set $\mathbf{y}(x) := \begin{bmatrix} P_m(x) \\ Y(x)Q_n(x) \end{bmatrix}$ and define Casorati determinants $D_i(x)$ by

$$(2.7) \quad D_1(x) := \det[\mathbf{y}(x), \mathbf{y}(qx)], \quad D_2(x) := \det[\mathbf{y}(x), \overline{\mathbf{y}}(x)], \quad D_3(x) := \det[\mathbf{y}(qx), \overline{\mathbf{y}}(x)].$$

Then the expressions (2.6) can be rewritten as follows:

$$(2.8) \quad \begin{aligned} L_2(x) &\propto D_1(x)\overline{y}(x) - D_2(x)y(qx) + D_3(x)y(x), \\ L_3(x) &\propto \overline{D}_1(x/q)y(x) + D_3(x/q)\overline{y}(x) - D_2(x)\overline{y}(x/q). \end{aligned}$$

Define basic quantities $G(x)$, $K(x)$ and $H(x)$ (e.g. (3.5), (3.17)) by

$$(2.9) \quad G(x) := Y(qx)/Y(x), \quad K(x) := \overline{Y}(x)/Y(x), \quad H(x) := \text{L.C.M.}(G_{\text{den}}(x), K_{\text{den}}(x)).$$

Here $G_{\text{den}}(x)$ and $G_{\text{num}}(x)$ are defined as the polynomials of the denominator and the numerator in $G(x)$ respectively, and $K_{\text{den}}(x)$ and $K_{\text{num}}(x)$ are similarly defined. For example in case of q - $E_6^{(1)}$, $G_{\text{den}}(x) = \prod_{i=1}^3 (1 - a_i x)$, $G_{\text{num}}(x) = \prod_{i=3}^3 (1 - b_i x)$, $K_{\text{den}}(x) = 1 - a_1 x$, $K_{\text{num}}(x) = 1 - b_1 x$ (see eq.(3.5)). Substituting these quantities into the determinants (2.7), we obtain the following expressions:

$$(2.10) \quad \begin{aligned} D_1(x) &= \frac{Y(x)}{G_{\text{den}}(x)} \{G_{\text{num}}(x)P_m(x)Q_n(qx) - G_{\text{den}}(x)P_m(qx)Q_n(x)\}, \\ D_2(x) &= \frac{Y(x)}{K_{\text{den}}(x)} \{K_{\text{num}}(x)P_m(x)\overline{Q}_n(x) - K_{\text{den}}(x)\overline{P}_m(x)Q_n(x)\}, \\ D_3(x) &= \frac{Y(x)}{H(x)} \left\{ \frac{H(x)}{K_{\text{den}}(x)} K_{\text{num}}(x)P_m(qx)\overline{Q}_n(x) - \frac{H(x)}{G_{\text{den}}(x)} G_{\text{num}}(x)\overline{P}_m(x)Q_n(qx) \right\}. \end{aligned}$$

Using the approximation condition (2.1) and the form of the basic quantities $G(x)$, $K(x)$, $H(x)$ (e.g. eqs.(3.5), (3.17)), we can investigate positions of zeros (e.g. $x = 0$) and degrees of the polynomials (e.g. $G_{\text{num}}(x)P_m(x)Q_n(qx) - G_{\text{den}}(x)P_m(qx)Q_n(x)$) within braces $\{ \}$ of the expressions (2.10). Then we can simply compute the determinants $D_i(x)$ (e.g. eqs.(3.6), (3.18)) except for some factors such as $1 - fx$, $1 - x/g$ and c_i in $D_i(x)$, where f , g and c_i are constants with respect to x (see Remark 6). In this way we obtain the contiguity relations $L_2 = 0$ and $L_3 = 0$ (e.g. eqs.(3.7), (3.19)).

Remark 5. On the gauge invariance of the product $C_0 C_1$

When the common normalization factor of the approximating polynomials $P_m(x)$ and $Q_n(x)$ is changed, an x -independent gauge transformation of $y(x)$ is induced in the contiguity relations $L_2 = 0$ and $L_3 = 0$. Under the x -independent gauge transformation of $y(x)$: $y(x) \mapsto Gy(x)$, the coefficients of $\overline{y}(x)$, $y(x/q)$, $y(x)$ and $y(x)$, $\overline{y}(x)$, $\overline{y}(x/q)$ in $L_2 = 0$ and $L_3 = 0$ (2.8) change as follows:

$$(2.11) \quad \begin{aligned} (D_1(x) : D_2(x) : D_3(x)) &\mapsto (\overline{G}D_1(x)/G : D_2(x) : D_3(x)) \\ (\overline{D}_1(x/q) : D_3(x/q) : D_2(x)) &\mapsto (G\overline{D}_1(x/q)/\overline{G} : D_3(x/q) : D_2(x)). \end{aligned}$$

The coefficients C_0 and C_1 in $L_2 = 0$ and $L_3 = 0$ (e.g. eqs.(3.7), (3.19)) are defined as the normalization factors of the coefficients of $\bar{y}(x)$ and $y(x)$ respectively. Then C_0 and C_1 change under the gauge transformation, but the product C_0C_1 is a gauge invariant quantity. Moreover C_0 and C_1 do not appear in the final form of the q -Painlevé equations (e.g. eqs.(3.8), (3.20)). \square

Remark 6. On two meanings of variables f, g and parameters m, n

We use f and g with two different meanings. The first meaning is constants (i.e. special solutions) f and g which are explicitly determined in terms of parameters a_i, b_i, m and n by the Padé approximation problem (e.g. eqs.(3.6), (3.7), (3.12), (3.18), (3.19), (3.24)). The second meaning is generic variables (i.e. generic solutions) f and g apart from the Padé approximation problem (e.g. eqs.(3.8), (3.11), (3.20), (3.23)), namely f and g are unknown functions in the q -Painlevé equation. In the items (c), (d) (resp. in the items (b), (e)) we consider f and g in the second meaning (resp. in the first meaning).

Similarly we use m and n with two meanings. In the first meaning m and $n \in \mathbb{Z}_{\geq 0}$ are integer parameters (e.g. eqs.(3.3), (3.6), (3.7), (3.15), (3.18), (3.19)). In the second meaning m and $n \in \mathbb{C}^\times$ are generic complex parameters, namely q^m and q^n are replaced by generic parameters a_0 and b_0 respectively (e.g. eqs.(3.8), (3.11), (3.20), (3.23)). In the items (c), (d) (resp. in the items (a), (b), (e)) we consider f and g in the second meaning (resp. in the first meaning). Then the result of the compatibility of the contiguity relations $L_2 = 0$ and $L_3 = 0$ also holds with respect to the second meaning. \square

2.3. (c) The q -Painlevé equation.

Let us consider generic variables f, g and generic parameter a_0, b_0 as in the second meaning in Remark 6, we can derive the q -Painlevé equation as the necessary condition for the compatibility of the contiguity relations $L_2 = 0$ and $L_3 = 0$ (e.g. eqs.(3.7), (3.19)). Computing the compatibility condition, we determine three variables \underline{g}, \bar{f} and C_0C_1 . Expressions for variables \underline{g} and \bar{f} are obtained in terms of variables f and g . An expression for the product C_0C_1 is obtained in terms of variables f, g and \bar{f} (and hence in terms of variables f and g).

The first and the second expressions are the q -Painlevé equation (e.g. eqs.(3.8), (3.20)). The third expression is a constraint for the product C_0C_1 (e.g. eqs.(3.9), (3.21)).

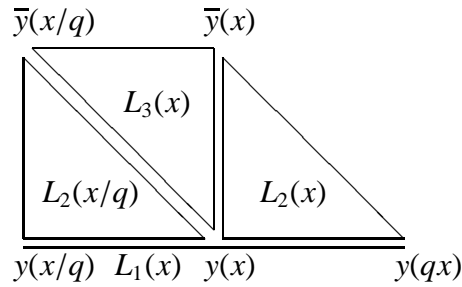
2.4. (d) The Lax pair.

Let us consider two linear three term equations for the unknown function $y(x)$: $L_1(x) = 0$ between $y(qx), y(x), y(x/q)$ and $L_2(x) = 0$ between $y(x), y(qx), \bar{y}(x)$, where L_1 and L_2 are given as expressions

$$(2.12) \quad \begin{aligned} L_1(x) &= A_1(x)y(x/q) + A_2(x)y(x) + A_3(x)y(qx), \\ L_2(x) &= A_4(x)\bar{y}(x) + A_5(x)y(x) + A_6(x)y(qx). \end{aligned}$$

The linear three term equations $L_1 = 0$ and $L_2 = 0$ (2.12) are called the "scalar Lax pair", when the compatibility condition of the linear equations $L_1 = 0$ and $L_2 = 0$ (2.12) is equivalent to a q -Painlevé equation.

We present how to compute the scalar Lax pair. Similarly to the item (c), let us consider generic variables f, g and generic parameter a_0, b_0 as in the second meaning in Remark 6. The Lax pair $L_1 = 0$ and $L_2 = 0$, which satisfies the compatibility condition, is derived using the results of the items (a)–(c) as follows: The Lax equation $L_2 = 0$ (e.g. eqs.(3.11), (3.23)) in the item (d) is the same as the contiguity relation $L_2 = 0$ (e.g. eqs.(3.7), (3.19)) in the item (b) under an x -independent gauge transform of $y(x)$ and changes of parameters. We can obtain the Lax equation $L_1 = 0$ as follows: First combining the contiguity relations $L_2 = 0$ and $L_3 = 0$ (e.g. eqs.(3.7), (3.19)) under generic variables f, g and generic parameters a_0, b_0 , one obtains a linear equation between the three terms $y(qx), y(x)$ and $y(x/q)$ (See the figure below), whose coefficient functions depend on the variables f, g, \bar{f}, C_0 and C_1 . However the variables C_0 and C_1 appear through the product $C_0 C_1$. Therefore expressing \bar{f} (e.g. eqs.(3.8), (3.20)) and $C_0 C_1$ (e.g. eqs.(3.9), (3.21)) in terms of f and g only, one obtains the Lax equation $L_1 = 0$ (e.g. eqs.(3.11), (3.23)).



Then, under generic variables f, g and generic parameters a_0, b_0 , the q -Painlevé equation (e.g. eqs.(3.8), (3.20)) is sufficient for the compatibility of the Lax pair $L_1 = 0$ and $L_2 = 0$ (e.g. eqs.(3.11), (3.23)). The similar proofs have already been given in [8, 31, 32].

2.5. (e) Special solutions.

By construction, expressions for constants f and g as in the first meaning in Remark 6 give a special solution for the q -Painlevé equation. We present how to compute determinant formulae of the special solutions f and g .

We derive the formulae (2.14) which are convenient for computing the special solutions f and g . We can assume the power series

$$(2.13) \quad Y(x) = \sum_{k=0}^{\infty} p_k x^k, \quad p_0 = 1, \quad p_i = 0 \quad (i < 0)$$

since the generating functions $Y(x)$ in Table (2.2) are holomorphic near $x = 0$. Then for each type of the functions $Y(x)$, the coefficients p_k (e.g. eqs.(3.25), (3.36)) are given respectively as the terminating cases of q -hypergeometric functions defined by eq.(2.4).

For a given function $Y(x)$, the polynomials $P_m(x)$ and $Q_n(x)$ of degree m and n for the approximation condition (2.1) are given by the following determinant expressions:

$$(2.14) \quad P_m(x) = \sum_{i=0}^m s_{(m^m, i)} x^i, \quad Q_n(x) = \sum_{i=0}^n s_{((m+1)^i, m^n-i)} (-x)^i$$

where s_λ is the Schur function defined by the Jacobi-Trudi formula

$$(2.15) \quad s_{(\lambda_1, \dots, \lambda_l)} := \det(p_{\lambda_i - i + j})_{i,j=1}^l.$$

We show the derivation of the expressions (2.14) as follows: The approximating polynomial $Q_n(x)$ satisfying the condition (2.1) can be given as the second expression of eq.(2.14) by Cramer's rule. The approximating polynomial $P_m(x)$ satisfying the condition (2.1) is given by the following computation: By using the relation

$$(2.16) \quad x^n Y(x) = \sum_{k=0}^{\infty} p_k x^{k+n} = \sum_{k=0}^{\infty} p_{k-n} x^k,$$

we have

$$(2.17) \quad Y(x)Q_n(x) = \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{m-n+1} & \cdots & p_m & p_{m+1} \\ x^n Y(x) & \cdots & xY(x) & Y(x) \end{vmatrix} = \sum_{k=0}^{\infty} \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{m-n+1} & \cdots & p_m & p_{m+1} \\ p_{k-n} & \cdots & p_{k-1} & p_k \end{vmatrix} x^k = \sum_{k=0}^{\infty} s_{(m^n, k)} x^k.$$

Here we note that

$$(2.18) \quad \sum_{k=m+1}^{m+n} s_{(m^n, k)} x^k = 0.$$

Substituting the relation (2.18) into the expression (2.17), we obtain

$$(2.19) \quad Y(x)Q_n(x) = \left(\sum_{k=0}^m + \sum_{k=m+n+1}^{\infty} \right) s_{(m^n, k)} x^k.$$

Hence the desired polynomial $P_m(x)$ is given as the first expression of the formulae (2.14).

Furthermore the polynomials $P_m(x)$ and $Q_n(x)$ in the formulae (2.14) can be expressed in terms of a single determinant as

$$(2.20) \quad P_m(x) = x^m s_{(m^{n+1})} |_{p_i \rightarrow \sum_{j=0}^i x^{-j} p_{i-j}}, \quad Q_n(x) = (-x)^n s_{((m+1)^n)} |_{p_i \rightarrow p_i - x^{-1} p_{i-1}}.$$

The formulae (2.14) and (2.20) have already appeared in [30].

Then we apply the general results described above to the case $N = 1$ (3.15) and $N = 2$ (3.3) of the function $\psi(x) := \prod_{i=1}^{N+1} \frac{(a_i x)_{\infty}}{(b_i x)_{\infty}}$, which can be written as

$$(2.21) \quad \psi(x) = \exp \left(\sum_{k=1}^{\infty} \sum_{s=1}^{N+1} \frac{b_s^k - a_s^k}{k(1 - q^k)} x^k \right) = \sum_{k=0}^{\infty} p_k x^k.$$

We note that this kind of expression (2.21) has already appeared in [25].

We show the method of computation of the special solutions f and g .

The expressions for the special solutions f and g can be derived by comparing the determinants $D_i(x)$ in eq.(2.10) and $D_i(x)$ (e.g. eqs.(3.6), (3.18)) in the item (b) as the identity with respect to the variable x and applying the formulae (2.14) and (2.20).

For example the computation for the case $q-E_6^{(1)}$ is as follows: Substituting $x = 1/a_i$ ($i = 1, 2$) into the determinants $D_1(x)$ in eq.(2.10) and $D_1(x)$ in (3.6) respectively, we obtain an expression for the special solution f in the first equation of eq.(3.12) by comparing the two expressions for $D_1(x)$ and applying the formulae (2.20). Similarly substituting $x = 1/b_i$ ($i = 2, 3$) into the determinants $D_3(x)$ in eq.(2.10) and $D_3(x)$ in eq.(3.6) respectively, we obtain an expression for the special solution g in the second equation of eq.(3.12) by comparing the two expressions for

$D_3(x)$ and applying the formulae (2.20).

3. MAIN RESULTS

In this section for each case q - $E_6^{(1)}$, q - $D_5^{(1)}$, q - $A_4^{(1)}$ and q -($A_2 + A_1$) $^{(1)}$, we present the results obtained through the method, which was explained in Section 2.

We use the following notations:

$$(3.1) \quad \begin{aligned} a_1 a_2 \cdots a_n / b_1 b_2 \cdots b_n &:= \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n}, \\ \tau_{m,n} &:= s_{(m^n)}, \end{aligned}$$

$$T_{a_i}(F) := F|_{a_i \rightarrow qa_i}, \quad T_{a_i}^{-1}(F) := F|_{a_i \rightarrow a_i/q}$$

for any quantity (or function) F depending on variables a_i and b_i , and by definition (2.15), the Schur function $s_{(m^n)}$ is expressed as

$$(3.2) \quad s_{(m^n)} = \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n-1} \\ p_{m-1} & \ddots & & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ p_{m-n+1} & \cdots & \cdots & p_m \end{vmatrix}$$

where the element p_k is defined in the power series (2.13).

3.1. Case q - $E_6^{(1)}$.

(a) Setting of the Padé approximation problem

In Table (2.2) the generating function and the constraint are established as

$$(3.3) \quad Y(x) := \frac{(a_1 x, a_2 x, a_3 x; q)_\infty}{(b_1 x, b_2 x, b_3 x; q)_\infty}, \quad \frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1,$$

and in Table (2.5) the time evolution is chosen as

$$(3.4) \quad T : (a_1, a_2, a_3, b_1, b_2, b_3, m, n) \mapsto (qa_1, a_2, a_3, qb_1, b_2, b_3, m, n).$$

(b) Contiguity relations

By the definition (2.9) we have the basic quantities

$$(3.5) \quad G(x) = \prod_{i=1}^3 \frac{(1 - b_i x)}{(1 - a_i x)}, \quad K(x) = \frac{1 - b_1 x}{1 - a_1 x}, \quad H(x) = \prod_{i=1}^3 (1 - a_i x),$$

and by the expression (2.10) we obtain the Casorati determinants

$$(3.6) \quad \begin{aligned} D_1(x) &:= c_0 (1 - x) x^{m+n+1} Y(x) / \prod_{i=1}^3 (1 - a_i x), \quad D_2(x) := c_1 x^{m+n+1} Y(x) / (1 - a_1 x), \\ D_3(x) &:= c_1 \frac{a_2 a_3 q^m g}{b_1} (1 - b_1 x) (1 - x/g) x^{m+n+1} Y(x) / \prod_{i=1}^3 (1 - a_i x) \end{aligned}$$

where f, g, c_0 and c_1 are constants depending on parameters $a_i, b_i \in \mathbb{C}^\times (i = 1, 2, 3)$, $m, n \in \mathbb{Z}_{\geq 0}$ but independent of x . Then the contiguity relations $L_2 = 0$ and $L_3 = 0$ are expressed by (3.7)

$$\begin{aligned} L_2(x) &= C_0(1 - xf)\bar{y}(x) - (1 - a_2x)(1 - a_3x)y(qx) + \frac{a_2a_3q^mg}{b_1}(1 - b_1x)(1 - x/g)y(x), \\ L_3(x) &= C_1(1 - x\bar{f}/q)y(x) + \frac{a_2a_3q^mg}{b_1}(1 - a_1x)(1 - x/qg)\bar{y}(x) - q^{m+n+1}(1 - b_2x/q)(1 - b_3x/q)\bar{y}(x/q) \end{aligned}$$

where $C_0 = c_0/c_1$ and $C_1 = \bar{c}_0/c_1$.

Take note that in the items (c) and (d) below we study the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.7) for generic complex parameters $a_0 = q^m$, $b_0 = q^n$ ($m, n \in \mathbb{C}^\times$) and generic variables f, g (depending on parameters $a_i, b_i \in \mathbb{C}^\times, i = 0, 1, 2, 3$) apart from the Padé approximation problem (2.1) with eqs.(3.3) and (3.4). (see Remark 6)

(c) The q -Painlevé equation

Compatibility of the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.7) gives the evolution equations and the constraint on the product C_0C_1 as follows:

$$\begin{aligned} (fg - 1)(f\underline{g} - 1) &= \frac{b_0b_1^2}{a_0a_2^2a_3^2} \frac{(f - a_2)(f - a_3)(f - b_2)(f - b_3)}{(f - a_1)(f - b_1)}, \\ (fg - 1)(\bar{f}g - 1) &= qa_1b_1 \frac{(g - 1/a_2)(g - 1/a_3)(g - 1/b_2)(g - 1/b_3)}{(g - b_1/a_0a_2a_3)(g - qb_0b_1/a_2a_3)} \end{aligned} \quad (3.8)$$

and

$$C_0C_1 = a_0(b_1 - a_0a_2a_3g)(qb_0b_1 - a_2a_3g)/b_1^2. \quad (3.9)$$

The evolution equations (3.8) are equivalent to the q -Painlevé equation of type $E_6^{(1)}$ given in [4, 6, 8, 13, 19]. The 8 singular points in coordinates (f, g) are on the two lines $f = \infty$ and $g = \infty$ and one curve $fg = 1$ as follows:

$$(f, g) = (1/a_2, a_2), (1/a_3, a_3), (1/b_2, b_2), (1/b_3, b_3), \\ (a_1, \infty), (b_1, \infty), (\infty, b_1/a_0a_2a_3), (\infty, qb_0b_1/a_2a_3). \quad (3.10)$$

(d) The Lax pair

The contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.7) give two scalar Lax equations $L_1 = 0$ and $L_2 = 0$ expressed by

$$\begin{aligned} (3.11) \quad L_1(x) &= \frac{a_0b_0(1 - b_1x/q)(1 - b_2x/q)(1 - b_3x/q)}{(1 - fx/q)} \left[y(x/q) - \frac{a_1(1 - a_2x/q)(1 - a_3x/q)}{b_0b_2b_3(1 - b_1x/q)(g - x/q)} y(x) \right] \\ &\quad + \frac{(1 - a_1x)(1 - a_2x)(1 - a_3x)}{q(1 - fx)} \left[y(qx) - \frac{b_0b_2b_3(1 - b_1x)(g - x)}{a_1(1 - a_2x)(1 - a_3x)} y(x) \right] \\ &\quad + \frac{a_0a_1}{b_2b_3g} \left[\left(1 - \frac{b_2b_3g}{qa_0a_1} \right) \left(1 - \frac{b_0b_2b_3g}{a_1} \right) + \frac{x(1 - a_2g)(1 - a_3g)(1 - b_2g)(1 - b_3g)}{(1 - fg)(gq - x)} \right] y(x), \\ L_2(x) &= (1 - xf)\bar{y}(x) - (1 - a_2x)(1 - a_3x)y(qx) + \frac{a_0a_2a_3g}{b_1}(1 - b_1x)(1 - x/g)y(x). \end{aligned}$$

The scalar Lax pair $L_1 = 0$ and $L_2 = 0$ (3.11) is expected to be equivalent to the 2×2 matrix ones in [23, 29] and the scalar ones in [4, 13, 32] by using suitable gauge transformations of $y(x)$. (Note that there are some typographical errors in eqs.(30) and (31) in [4], namely the expressions $(b_4 q' - z)$ and $Y(x)$ should read $(b_4 q' - z)t^2$ and $Y(z)$ respectively.)

(e) Special solutions

The determinant formulae of the special solutions are given as

$$(3.12) \quad \begin{aligned} \frac{1 - f/a_1}{1 - f/a_2} &= \frac{a_1 \prod_{i=1}^3 (1 - b_i/a_1) T_{a_1}(\tau_{m,n+1}) T_{a_1}^{-1}(\tau_{m+1,n})}{a_2 \prod_{i=1}^3 (1 - b_i/a_2) T_{a_2}(\tau_{m,n+1}) T_{a_2}^{-1}(\tau_{m+1,n})}, \\ \frac{1 - 1/b_2 g}{1 - 1/b_3 g} &= \frac{b_2 \prod_{i=2}^3 (1 - a_i/b_2) T_{b_2}^{-1}(\tau_{m,n+1}) T_{b_2}(\bar{\tau}_{m+1,n})}{b_3 \prod_{i=2}^3 (1 - a_i/b_3) T_{b_3}^{-1}(\tau_{m,n+1}) T_{b_3}(\bar{\tau}_{m+1,n})}. \end{aligned}$$

Here the element p_k in the determinant $\tau_{m,n}$ (3.1) is given by

$$(3.13) \quad p_k = \frac{b_3^k \left(\frac{a_3}{b_3}; q \right)_k}{(q; q)_k} \varphi_D^{(2)} \left(q^{-k}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, q^{-k+1} \frac{b_3}{a_3}; q \frac{b_1}{a_3}, q \frac{b_2}{a_3} \right)$$

and $\varphi_D^{(l)}$ is the q -Appell Lauricella function (i.e. the multivariable q -hypergeometric function) [2] defined by

$$(3.14) \quad \varphi_D^{(l)}(\alpha, \beta_1, \dots, \beta_l, \gamma; z_1, \dots, z_l) := \sum_{m_i \geq 0} \frac{(\alpha)_{|m|} (\beta_1)_{m_1} \dots (\beta_l)_{m_l}}{(\gamma)_{|m|} (q)_{m_1} \dots (q)_{m_l}} z_1^{m_1} \dots z_l^{m_l}$$

where $|m| = m_1 + \dots + m_l$.

Remark 7. On the transformation between the terminating $\varphi_D^{(2)}$ and the terminating ${}_3\varphi_2$

The element p_k (3.13) expressed by the terminating q -Appell Lauricella series $\varphi_D^{(2)}$ is rewritten in terms of the terminating q -hypergeometric series ${}_3\varphi_2$ (2.4) (big q -Jacobi polynomials [6, 9, 16]) under the transformation of $b_3 = q^{k-1} a_1 a_2$ and $a_3 = b_1 b_2$. \square

These determinant formulae of the q -hypergeometric solutions (3.12) are expected to be equivalent to those in [4, 13]. (Note that there is a typographical error in eq.(38) in [4], namely the expression $T_{a_2} T_{a_3}(\tau_{m,n-1})$ should read $T_{a_2} T_{a_4}(\tau_{m,n-1})$.) Determinant formulae of q -hypergeometric solutions still have not been given in terms of the non-terminating q -hypergeometric series ${}_3\varphi_2$ as far as we know.

3.2. Case q - $D_5^{(1)}$.

The contents of these subsections (a), (b), (c) are the same as [4].

(a) Setting of the Padé approximation problem

In Table (2.2) the generating function is established as

$$(3.15) \quad Y(x) := \frac{(a_1 x, a_2 x; q)_\infty}{(b_1 x, b_2 x; q)_\infty}$$

and in Table (2.5) the time evolution is chosen as

$$(3.16) \quad T : (a_1, a_2, b_1, b_2, m, n) \mapsto (qa_1, a_2, qb_1, b_2, m, n).$$

(b) Contiguity relations

By the definition (2.9) we have the basic quantities

$$(3.17) \quad G(x) = \prod_{i=1}^2 \frac{(1 - b_i x)}{(1 - a_i x)}, \quad K(x) = \frac{1 - b_1 x}{1 - a_1 x}, \quad H(x) = \prod_{i=1}^2 (1 - a_i x),$$

and by the expression (2.10) we obtain the Casorati determinants

$$(3.18) \quad D_1(x) =: \frac{c_0(1 - xf)x^{m+n+1}Y(x)}{\prod_{i=1}^2 (1 - a_i x)}, \quad D_2(x) =: \frac{c_1 x^{m+n+1}Y(x)}{1 - a_1 x}, \quad D_3(x) =: \frac{c_2(1 - b_1 x)x^{m+n+1}Y(x)}{\prod_{i=1}^2 (1 - a_i x)}$$

where f, c_0, c_1 and c_2 are constants depending on parameters $a_1, a_2, b_1, b_2 \in \mathbb{C}^\times$, $m, n \in \mathbb{Z}_{\geq 0}$ but independent of x . Then the contiguity relations $L_2 = 0$ and $L_3 = 0$ are expressed by

$$(3.19) \quad \begin{aligned} L_2(x) &= C_0(1 - xf)\bar{y}(x) - (1 - a_2 x)y(qx) + (1 - b_1 x)y(x)/g, \\ L_3(x) &= C_1(1 - xf/q)y(x) + (1 - a_1 x)\bar{y}(x)/g - q^{m+n+1}(1 - b_2 x/q)\bar{y}(x/q) \end{aligned}$$

where $C_0 = c_0/c_1$, $C_1 = \bar{c}_0/c_1$ and $g = c_1/c_2$.

Take note that in the items (c) and (d) we study the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.19) for generic complex parameters $a_0 = q^m$, $b_0 = q^n$ ($m, n \in \mathbb{C}^\times$) and generic variables f, g (depending on parameters $a_i, b_i \in \mathbb{C}^\times$, $i = 0, 1, 2$) apart from the Padé approximation problem (2.1) with eqs.(3.15) and (3.16). (see Remark 6)

(c) The q -Painlevé equation

Compatibility of the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.19) gives the evolution equations and the constraint on the product $C_0 C_1$ as follows:

$$(3.20) \quad g\underline{g} = \frac{1}{qa_0 b_0} \frac{(f - a_1)(f - b_1)}{(f - a_2)(f - b_2)}, \quad f\bar{f} = a_2 b_2 \frac{(g - b_1/a_0 a_2)(g - a_1/b_0 b_2)}{(g - 1)(g - 1/qa_0 b_0)}$$

and

$$(3.21) \quad C_0 C_1 = (1 - g)(1 - qa_0 b_0 g)/g^2.$$

The evolution equations (3.20) are equivalent to the q -Painlevé equation of type $D_5^{(1)}$ given in [4, 5, 6, 8, 13]. The 8 singular points in coordinates (f, g) are on the four lines $f = 0$, $f = \infty$, $g = 0$ and $g = \infty$ as follows:

$$(3.22) \quad (f, g) = (a_1, 0), (b_1, 0), (0, b_1/a_0 a_2), (0, a_1/b_0 b_2), (\infty, 1), (\infty, 1/qa_0 b_0), (a_2, \infty), (b_2, \infty).$$

(d) The Lax pair

The contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.19) give two scalar Lax equations $L_1 = 0$ and $L_2 = 0$ expressed by

$$\begin{aligned}
 (3.23) \quad L_1(x) &= \left[(1-g)(1-qa_0b_0g) - \frac{x(a_1-b_0b_2g)(b_1-a_0a_2g)}{f} \right] y(x) \\
 &+ \frac{g(1-a_1x)(1-a_2x)}{1-fx} \left[y(qx) - \frac{1-b_1x}{g(1-a_2x)} y(x) \right] \\
 &+ \frac{qa_0b_0g(1-b_1x/q)(1-b_2x/q)}{1-fx/q} \left[y(x/q) - \frac{g(1-a_2x/q)}{1-b_1x/q} y(x) \right], \\
 L_2(x) &= (1-xf)\bar{y}(x) - (1-a_2x)y(qx) + (1-b_1x)y(x)/g.
 \end{aligned}$$

The scalar Lax pair $L_1 = 0$ and $L_2 = 0$ (3.23) is equivalent to the 2×2 matrix ones in [5, 18] and the scalar one in [13, 32] by using suitable gauge transformations of $y(x)$.

(e) Special solutions

The determinant formulae of the special solutions are given as

$$\begin{aligned}
 (3.24) \quad \frac{1-f/a_1}{1-f/a_2} &= \frac{a_1 \prod_{i=1}^2 (1-b_i/a_1) T_{a_1}(\tau_{m,n+1}) T_{a_1}^{-1}(\tau_{m+1,n})}{a_2 \prod_{i=1}^2 (1-b_i/a_2) T_{a_2}(\tau_{m,n+1}) T_{a_2}^{-1}(\tau_{m+1,n})}, \\
 g &= \frac{a_1(1-b_1/a_1) T_{a_1}(\tau_{m,n+1}) T_{a_1}^{-1}(\bar{\tau}_{m+1,n})}{q^n a_2(1-b_2/a_2) T_{a_2}(\bar{\tau}_{m,n+1}) T_{a_2}^{-1}(\tau_{m+1,n})}
 \end{aligned}$$

where the element p_k in the determinant $\tau_{m,n}$ (3.1) is given by

$$(3.25) \quad p_k = b_2^k \frac{(a_2/b_2; q)_k}{(q; q)_k} {}_2\varphi_1 \left(\begin{matrix} q^{-k}, a_1/b_1 \\ b_2 q^{-k+1}/a_2 \end{matrix}; q, \frac{b_1}{a_2} q \right).$$

The element p_k (3.25) is expressed in terms of the terminating q -hypergeometric series ${}_2\varphi_1$ (2.4) (little q -Jacobi polynomials [6, 9, 16]). These determinant formulae of the q -hypergeometric solutions (3.24) are expected to be equivalent to those in [13] and the terminating case of those in [20].

3.3. Case q - $A_4^{(1)}$.

(a) Setting of the Padé approximation problem

In Table (2.2) the generating function is established as

$$(3.26) \quad Y(x) := \frac{(a_1x, a_2x; q)_\infty}{(b_1x; q)_\infty}$$

and in Table (2.5) the time evolution is chosen as

$$(3.27) \quad T : (a_1, a_2, b_1, m, n) \mapsto (qa_1, a_2, qb_1, m, n).$$

(b) Contiguity relations

By the definition (2.9) we have the basic quantities

$$(3.28) \quad G(x) = \frac{1 - b_1 x}{(1 - a_1 x)(1 - a_2 x)}, \quad K(x) = \frac{1 - b_1 x}{1 - a_1 x}, \quad H(x) = (1 - a_1 x)(1 - a_2 x),$$

and by the expression (2.10) we obtain the Casorati determinants

$$(3.29) \quad D_1(x) =: \frac{c_0(1 - xf)x^{m+n+1}Y(x)}{(1 - a_1 x)(1 - a_2 x)}, \quad D_2(x) =: \frac{c_1 x^{m+n+1}Y(x)}{1 - a_1 x}, \quad D_3(x) =: \frac{c_2 x^{m+n+1}Y(x)}{(1 - a_1 x)(1 - a_2 x)}$$

where f, c_0, c_1 and c_2 are constants depending on parameters $a_1, a_2, b_1 \in \mathbb{C}^\times$, $m, n \in \mathbb{Z}_{\geq 0}$ but independent of x . Then the contiguity relations $L_2 = 0$ and $L_3 = 0$ are expressed by

$$(3.30) \quad \begin{aligned} L_2(x) &= C_0(1 - xf)\bar{y}(x) - (1 - a_2 x)y(qx) + (1 - b_1 x)y(x)/g, \\ L_3(x) &= C_1(1 - x\bar{f}/q)y(x) + (1 - a_1 x)\bar{y}(x)/g - q^{m+n+1}\bar{y}(x/q) \end{aligned}$$

where $C_0 = c_0/c_1$, $C_1 = \bar{c}_0/c_1$ and $g = c_1/c_2$.

Take note that in the items (c) and (d) below we study the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.30) for generic complex parameters $a_0 = q^m$, $b_0 = q^n$ ($m, n \in \mathbb{C}^\times$) and generic variables f, g (depending on parameters $a_0, a_1, a_2, b_0, b_1 \in \mathbb{C}^\times$) apart from the Padé approximation problem (2.1) with eqs.(3.26) and (3.27). (see Remark 6)

(c) The q -Painlevé equation

Compatibility of the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.30) gives the evolution equations and the constraint on the product $C_0 C_1$ as follows:

$$(3.31) \quad g\bar{g} = \frac{1}{qa_0 b_0} \frac{(f - a_1)(f - b_1)}{f(f - a_2)}, \quad f\bar{f} = -\frac{a_1 a_2}{b_0} \frac{g - b_1/a_0 a_2}{(g - 1)(g - 1/qa_0 b_0)}.$$

and

$$(3.32) \quad C_0 C_1 = (1 - g)(1 - qa_0 b_0 g)/g^2.$$

The evolution equations (3.31) are equivalent to the q -Painlevé equation of type $A_4^{(1)}$ given in [6, 8, 10, 13]. The 8 singular points in coordinates (f, g) are on the four lines $f = 0$, $f = \infty$, $g = 0$ and $g = \infty$ as follows:

$$(3.33) \quad (f, g) = (a_1, 0), (b_1, 0), (0, b_1/a_0 a_2), (\infty, 1), (\infty, 1/qa_0 b_0), (a_2, \infty), (\epsilon, -a_1/b_0 \epsilon)_2$$

where the last point is a double point at $(0, \infty)$ with the gradient $fg = -a_1/b_0$. (The meaning of the double point is also written in [8].)

(d) The Lax pair

The contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.30) give two scalar Lax equations $L_1 = 0$ and $L_2 = 0$ expressed by

$$(3.34) \quad \begin{aligned} L_1(x) &= \frac{qa_0b_0g(1-b_1x/q)}{fx/q-1} \left[y(x/q) - \frac{g(1-a_2x/q)}{1-b_1x/q} y(x) \right] \\ &+ \frac{g(1-a_1x)(1-a_2x)}{fx-1} \left[y(qx) - \frac{1-b_1x}{g(1-a_2x)} y(x) \right] \\ &+ \left[\frac{a_1(b_1-a_0a_2g)x}{f} - (g-1)(qa_0b_0-1) \right] y(x), \\ L_2(x) &= (1-xf)\bar{y}(x) - (1-a_2x)y(qx) + (1-b_1x)y(x)/g. \end{aligned}$$

The scalar Lax pair $L_1 = 0$ and $L_2 = 0$ (3.34) is equivalent to the 2×2 matrix one for the q -Painlevé equation of type q - $P(A_4)$ in [12] and the scalar one in [13] by using a suitable gauge transformation of $y(x)$.

(e) Special solutions

The determinant formulae of the special solutions are given as

$$(3.35) \quad \begin{aligned} \frac{1-f/a_1}{1-f/a_2} &= \frac{a_1(1-b_1/a_1)}{a_2(1-b_1/a_2)} \frac{T_{a_1}(\tau_{m,n+1})T_{a_1}^{-1}(\tau_{m+1,n})}{T_{a_2}(\tau_{m,n+1})T_{a_2}^{-1}(\tau_{m+1,n})}, \\ g &= \frac{a_1}{q^n a_2} \frac{T_{a_1}(\tau_{m,n+1})T_{a_1}^{-1}(\bar{\tau}_{m+1,n})}{T_{a_2}(\bar{\tau}_{m,n+1})T_{a_2}^{-1}(\tau_{m+1,n})} \end{aligned}$$

where the element p_k in the determinant $\tau_{m,n}$ (3.1) is given by

$$(3.36) \quad p_k = q^{\binom{k}{2}} \frac{(-a_2)^k}{(q; q)_k} {}_2\varphi_1 \left(\begin{matrix} q^{-k}, a_1/b_1 \\ 0 \end{matrix}; q, \frac{b_1}{a_2} q \right).$$

The element p_k (3.36) is expressed in terms of the terminating q -hypergeometric series ${}_2\varphi_1$ (2.4) (q -Laguerre polynomials [6, 9, 16]). These determinant formulae of the q -hypergeometric solutions (3.35) are expected to be equivalent to those in [13] and the terminating case of those in [3].

3.4. Case q -($A_2 + A_1$)⁽¹⁾.

(a) Setting of the Padé approximation problem

In Table (2.2) the generating function is established as

$$(3.37) \quad Y(x) := (a_1x, a_2x; q)_\infty$$

and in Table (2.5) the time evolution is chosen as

$$(3.38) \quad T : (a_1, a_2, m, n) \mapsto (qa_1, a_2, m, n).$$

(b) Contiguity relations

By the definition (2.9) we have the basic quantities

$$(3.39) \quad G(x) = \frac{1}{(1-a_1x)(1-a_2x)}, \quad K(x) = \frac{1}{1-a_1x}, \quad H(x) = (1-a_1x)(1-a_2x),$$

and by the expression (2.10) we obtain the Casorati determinants

(3.40)

$$D_1(x) =: \frac{c_0(1 - xf)x^{m+n+1}Y(x)}{(1 - a_1x)(1 - a_2x)}, \quad D_2(x) =: \frac{c_1x^{m+n+1}Y(x)}{1 - a_1x}, \quad D_3(x) =: \frac{c_2x^{m+n+1}Y(x)}{(1 - a_1x)(1 - a_2x)}$$

where f, c_0, c_1 and c_2 are constants depending on parameters $a_1, a_2 \in \mathbb{C}^\times$, $m, n \in \mathbb{Z}_{\geq 0}$ but independent of x . Then the contiguity relations $L_2 = 0$ and $L_3 = 0$ are expressed by

$$(3.41) \quad \begin{aligned} L_2(x) &= C_0(1 - xf)\bar{y}(x) - (1 - a_2x)y(qx) + y(x)/g, \\ L_3(x) &= C_1(1 - x\bar{f}/q)y(x) + (1 - a_1x)\bar{y}(x)/g - q^{m+n+1}\bar{y}(x/q) \end{aligned}$$

where $C_0 = c_0/c_1$, $C_1 = \bar{c}_0/c_1$ and $g = c_1/c_2$.

Take note that in the items (c) and (d) below we study the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.41) for generic complex parameters $a_0 = q^m$, $b_0 = q^n$ ($m, n \in \mathbb{C}^\times$) and generic variables f, g (depending on parameters $a_0, a_1, a_2, b_0 \in \mathbb{C}^\times$) apart from the Padé approximation problem (2.1) with eqs.(3.37) and (3.38). (see Remark 6)

(c) The q -Painlevé equation

Compatibility of the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.41) gives the evolution equations and the constraint on the product C_0C_1 as follows:

$$(3.42) \quad g\underline{g} = \frac{1}{qa_0b_0} \frac{f - a_1}{f - a_2}, \quad f\bar{f} = -\frac{a_1a_2}{b_0} \frac{g}{(g - 1)(g - 1/qa_0b_0)}$$

and

$$(3.43) \quad C_0C_1 = (1 - g)(1 - qa_0b_0g)/g^2.$$

The evolution equations (3.42) are equivalent to the q -Painlevé equation of type $(A_2 + A_1)^{(1)}$, namely P_{IV} , given in [7, 8, 13, 21]. The 8 singular points in coordinates (f, g) are on the four lines $f = 0, f = \infty, g = 0$ and $g = \infty$ as follows:

$$(3.44) \quad (f, g) = (a_1, 0), (\infty, 1), (\infty, 1/qa_0b_0), (a_2, \infty), (\epsilon, -a_1/b_0\epsilon)_2, (\epsilon, -\epsilon/a_0a_2)_2.$$

Here the fifth point is a double point at $(0, \infty)$ with the gradient $fg = -a_1/b_0$ and the sixth point is a double point at $(0, 0)$ with the gradient $g/f = -1/a_0a_2$. (The meaning of the two double points are also written in [8].)

(d) The Lax pair

The contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.41) give two scalar Lax equations $L_1 = 0$ and $L_2 = 0$ expressed by

$$(3.45) \quad \begin{aligned} L_1(x) &= \frac{qa_0b_0g}{1 - fx/q} [y(x/q) - g(1 - a_2x/q)y(x)] \\ &\quad + \frac{g(1 - a_1x)(1 - a_2x)}{1 - fx} \left[y(qx) - \frac{1}{g(1 - a_2x)} y(x) \right] \\ &\quad + \left[\frac{a_0a_1a_2gx}{f} + (g - 1)(qa_0b_0 - 1) \right] y(x), \\ L_2(x) &= (1 - xf)\bar{y}(x) - (1 - a_2x)y(qx) + y(x)/g. \end{aligned}$$

The scalar Lax pair $L_1 = 0$ and $L_2 = 0$ (3.45) is equivalent to the 2×2 matrix one for the q -Painlevé equation of type q - $P(A_5)^\#$ in [12] and the scalar one in [13] by using a suitable gauge transformation of $y(x)$.

(e) Special solutions

The determinant formulae of the special solutions are given as

$$(3.46) \quad \begin{aligned} \frac{1 - f/a_1}{1 - f/a_2} &= \frac{a_1 T_{a_1}(\tau_{m,n+1}) T_{a_1}^{-1}(\tau_{m+1,n})}{a_2 T_{a_2}(\tau_{m,n+1}) T_{a_2}^{-1}(\tau_{m+1,n})}, \\ g &= \frac{a_1 T_{a_1}(\tau_{m,n+1}) T_{a_1}^{-1}(\bar{\tau}_{m+1,n})}{q^n a_2 T_{a_2}(\bar{\tau}_{m,n+1}) T_{a_2}^{-1}(\tau_{m+1,n})} \end{aligned}$$

where the element p_k in the determinant $\tau_{m,n}$ (3.1) is given by

$$(3.47) \quad p_k = (-1)^k q^{\binom{k}{2}} \frac{(a_2; q)_k}{(q; q)_k} {}_1\varphi_1 \left(\begin{matrix} q^{-k} \\ 0 \end{matrix}; q, \frac{a_1}{a_2} q \right)$$

The element p_k (3.47) is expressed in terms of the terminating q -hypergeometric series ${}_2\varphi_1$ (2.4) (Stieltjes-Wigert polynomials [6, 9, 16]). These determinant formulae of the q -hypergeometric solutions (3.46) are expected to be equivalent to those in [13] and the terminating case of those in [14].

4. CONCLUSION

4.1. Summary.

In this paper for the generating function $Y(x)$ given in Table (2.2), we established the Padé approximation problem related to the q -Painlevé equations of type $E_6^{(1)}$, $D_5^{(1)}$, $A_4^{(1)}$ and $(A_2 + A_1)^{(1)}$. Then for the time evolution T given in Table (2.5), we established another Padé approximation problem. By solving these problems, we derived the evolution equations, the scalar Lax pairs and the determinant formulae of the special solutions for the corresponding q -Painlevé equations. The main results are given in Section 3.

4.2. Problems.

Some open problems related to the results of this paper are as follows:

1. In this paper by choosing certain time evolutions T , we applied the Padé method to each type q - $E_6^{(1)}$, q - $D_5^{(1)}$, q - $A_4^{(1)}$ and q - $(A_2 + A_1)^{(1)}$. Moreover by choosing other time evolutions, we can perform similar computations. It may be interesting to study the relation between the Padé approximation method for the various time evolutions and Bäcklund transformations of the affine Weyl group, for example the case q - $E_6^{(1)}$ of the interpolation problem in [4].

2. By the results of this paper, it turned out that the Padé approximation method could be applied to the q -Painlevé equations of type $E_6^{(1)}$, $D_5^{(1)}$, $A_4^{(1)}$ and $(A_2 + A_1)^{(1)}$. It may be interesting to study the degenerations between these results.

3. In this paper we applied the Padé method of differential grid (i.e. Padé approximation) to the q -Painlevé equations of type $E_6^{(1)}$, $D_5^{(1)}$, $A_4^{(1)}$ and $(A_2 + A_1)^{(1)}$. It may be interesting to study whether the Padé method can be also applied to additive difference Painlevé equations.

4. It may be interesting to study whether the Padé method can be further applied to the other generalized Painlevé systems, for example the q -Garnier system in [22] and the higher order q -Painlevé system in [24].

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REFERENCES

- [1] Clarkson P.A., *Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations*, J. Phys., A **46**, no.18, (2013), 185205–185222.
- [2] Gasper G., and Rahman M., *Basic Hypergeometric Series. With a foreword by Richard Askey. Second edition. Encyclopedia of Mathematics and its Applications*, Volume **96**, Cambridge University Press, Cambridge, (2004).
- [3] Hamamoto T., and Kajiwara K., *Hypergeometric solutions to the q -Painlevé equation of type $A_4^{(1)}$* , J. Phys. A: Math. Theor., **40** (2007), 12509–12524.
- [4] Ikawa Y., *Hypergeometric Solutions for the q -Painlevé Equation of Type $E_6^{(1)}$ by the Padé method*, Lett. Math. Phys., Volume **103**, Issue 7 (2013), 743–763.
- [5] Jimbo M., and Sakai H., *A q -analog of the sixth Painlevé equation*, Lett. Math. Phys., **38** (1996), 145–154.
- [6] Kajiwara K., Masuda T., Noumi M., Ohta Y., and Yamada Y., *Hypergeometric solutions to the q -Painlevé equations*, Int. Math. Res. Not. 2004, **47** (2004), 2497–2521.
- [7] Kajiwara K., Noumi M., and Yamada Y., *A study on the fourth q -Painlevé equation*, J. Phys. A: Math. Gen., **34** (2001), 8563–8581.
- [8] Kajiwara K., Noumi M., and Yamada Y., *Geometric aspects of Painlevé equations*, arXiv 1509.08186 [nlin.SI].
- [9] Koekoek R., and Swarttouw R. F., *The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue*, Delft University of Technology, Department of Technical Mathematics and Informatics Report (1998) no. 98–17.
- [10] Kruskal M. D., Tamizhmani K. M., Grammaticos B., and Ramani A., *Asymmetric discrete Painlevé equations*, Regul. Chaot. Dyn., **5** (2000), 273–281.
- [11] Magnus A., *Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials*, J. Comput. Appl. Math., **57** (1995), 215–237.
- [12] Murata M., *Lax forms of the q -Painlevé equations*, J. Phys. A: Math. Theor., **42** (2009), 115201–115217.
- [13] Nagao, H., *The Padé interpolation method applied to q -Painlevé equations*, Lett. Math. Phys. **105** (2015), no. 4, 503–521.
- [14] Nakazono N., *Hypergeometric tau Functions of the q -Painlevé Systems of Type $(A_2 + A_1)^{(1)}$* SIGMA, **6** (2010), 084–099.
- [15] Nakazono N., *Solutions to discrete Painlevé systems arising from two types of orthogonal polynomials*, Reports of RIAM Symposium, **23AO–S7** (2013), 35–41.
- [16] Noumi M., *Special functions arising from discrete Painlevé equations: a survey.*, J. Comput. Appl. Math. **202** (2007), no. 1, 48–55.
- [17] Noumi M., Tsujimoto S., and Yamada Y., *Padé interpolation for elliptic Painlevé equation*, Symmetries, integrable systems and representations, Springer Proc. Math. Stat., Volume **40** (2013), 463–482.

- [18] Ormerod C.M., Witte N.S., and Forrester P.J., *Connection preserving deformations and q -semi-classical orthogonal polynomials*, Nonlinearity, **24** (2011), 2405–2434.
- [19] Ramani A., Grammaticos B., Tamizhmani T., and Tamizhmani K.M., *Special Function Solutions of the Discrete Painlevé Equations*, Comput. Math. Appl., **42** (2001), no. 3–5, 603–614.
- [20] Sakai H., *Casorati determinant solutions for the q -difference sixth Painlevé equations*, Nonlinearity, **11** (1998), 823–833.
- [21] Sakai H., *Rational surfaces with affine root systems and geometry of the Painlevé equations*, Commun. Math. Phys., **220** (2001), 165–221.
- [22] Sakai H., *A q -Analog of the Garnier System*, Funkcial. Ekvac. **48** (2005), 273–297.
- [23] Sakai H., *Lax form of the q -Painlevé equation associated with the $A_2^{(1)}$ surface*, J. Phys. A: Math. Gen., **39** (2006), 12203–12210.
- [24] Suzuki T., *A q -analogue of the Drinfeld-Sokolov hierarchy of type A and q -Painlevé system*, AMS Contemp. Math. **651** (2015), 25–38.
- [25] Tsuda T., *On an integrable system of q -difference equations satisfied by the universal characters: its Lax formalism and an application to q -Painlevé equations*, Comm. Math. Phys. **293** (2010), 347–359.
- [26] Van W.A., *Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials. Difference equations, special functions and orthogonal polynomials*, World Sci. Publ., Hackensack, NJ (2007), 687–725.
- [27] Witte N.S., *Biorthogonal Systems on the Unit Circle, Regular Semiclassical Weights, and the Discrete Garnier Equations*, IMRN, **6**(2009), 988–1025.
- [28] Witte N.S., *Semiclassical orthogonal polynomial systems on nonuniform lattices, deformations of the Askey table, and analogues of isomonodromy*, Nagoya Mathematical Journal, **219** (2015), 127–234.
- [29] Witte N.S., and Ormerod C.M., *Construction of a Lax Pair for the $E_6^{(1)}$ q -Painlevé System*, SIGMA, **8** (2012), 097–123.
- [30] Yamada Y., *Padé method to Painlevé equations*, Funkcial. Ekvac., **52** (2009), 83–92.
- [31] Yamada Y., *A Lax formalism for the elliptic difference Painlevé equation*, SIGMA **5** (2009) 042 (15pp).
- [32] Yamada Y., *Lax formalism for q -Painlevé equations with affine Weyl group symmetry of type $E_n^{(1)}$* , IMRN, **17** (2011), 3823–3838.
- [33] Yamada Y., *A simple expression for discrete Painlevé equations*, RIMS Kokyuroku Bessatsu, **B47** (2014), 087–095.

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